

$W^{2,p}$ -A PRIORI ESTIMATES FOR THE NEUTRAL POINCARÉ PROBLEM

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To the memory of Filippo Chiarenza

ABSTRACT. A degenerate oblique derivative problem is studied for uniformly elliptic operators with low regular coefficients in the framework of Sobolev's classes $W^{2,p}(\Omega)$ for arbitrary $p > 1$. The boundary operator is prescribed in terms of a directional derivative with respect to the vector field ℓ that becomes tangential to $\partial\Omega$ at the points of some non-empty subset $\mathcal{E} \subset \partial\Omega$ and is directed outwards Ω on $\partial\Omega \setminus \mathcal{E}$. Under quite general assumptions of the behaviour of ℓ , we derive *a priori* estimates for the $W^{2,p}(\Omega)$ -strong solutions for any $p \in (1, \infty)$.

INTRODUCTION

The lecture deals with regularity in Sobolev's spaces $W^{2,p}(\Omega)$, $\forall p \in (1, \infty)$, of the strong solutions to the oblique derivative problem

$$(1) \quad \begin{cases} \mathcal{L}u := a^{ij}(x)D_{ij}u = f(x) & \text{a.e. } \Omega, \\ \mathcal{B}u := \partial u / \partial \ell = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

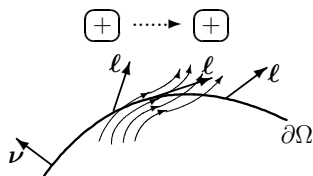
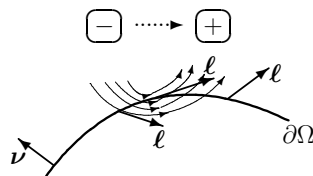
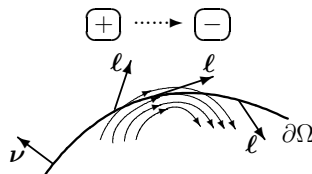
where \mathcal{L} is a uniformly elliptic operator with low regular coefficients and \mathcal{B} is prescribed in terms of a directional derivative with respect to the unit vector field $\ell(x) = (\ell^1(x), \dots, \ell^n(x))$ defined on $\partial\Omega$, $n \geq 3$. Precisely, we are interested in the Poincaré problem (1) (cf. [19, 22, 18]), that is, a situation when $\ell(x)$ becomes *tangential* to $\partial\Omega$ at the points of a non-empty subset \mathcal{E} of $\partial\Omega$.

From a mathematical point of view, (1) is *not* an elliptic boundary value problem. In fact, it follows from the general PDEs theory that (1) is a *regular (elliptic)* problem *if and only if* the Shapiro–Lopatinskij complementary condition is satisfied which means ℓ must be transversal to $\partial\Omega$ when $n \geq 3$ and $|\ell| \neq 0$ as $n = 2$. If ℓ is *tangent* to $\partial\Omega$ then (1) is a *degenerate* problem and new effects occur in contrast to the regular case. It turns out that the qualitative properties of (1) depend on the behaviour of ℓ near the set of tangency \mathcal{E} and especially on the way the normal component $\gamma\nu$ of ℓ (with respect to the outward normal ν to $\partial\Omega$) changes or no its sign on the trajectories of ℓ when these cross \mathcal{E} . The main results were obtained by Hörmander [6], Egorov and Kondrat'ev [2], Maz'ya [10], Maz'ya and Paneah [11], Melin and Sjöstrand [12], Paneah [17] and good surveys and details can be found in Popivanov and Palagachev [22] and Paneah [18]. The problem (1) has been studied in the framework of Sobolev spaces $H^s(\equiv H^{s,2})$ assuming C^∞ -smooth data and this naturally involved techniques from the pseudo-differential calculus.

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The simplest case arises when $\gamma := \ell \cdot \nu$, even if zero on \mathcal{E} , conserves the sign on $\partial\Omega$. Then \mathcal{E} and ℓ are of *neutral* type (a terminology coming from the physical interpretation of (1) in the theory of Brownian motion, see [22]) and (1) is a problem of Fredholm type (cf. [2]). Assume now that γ changes the sign from “−” to “+” in positive direction along the ℓ -integral curves passing through the points of \mathcal{E} . Then ℓ is of *emergent* type and \mathcal{E} is called *attracting* manifold. The new effect appearing now is that the *kernel* of (1) is *infinite-dimensional* ([6]) and to get a well-posed problem one has to modify (1) by prescribing the values of u on \mathcal{E} (cf. [2]). Finally, suppose the sign of γ changes from “+” to “−” along the ℓ -trajectories. Now ℓ is of *submergent* type and \mathcal{E} corresponds to a *repellent* manifold. The problem (1) has *infinite-dimensional cokernel* ([6]) and Maz’ya and Paneah [11] were the first to propose a relevant modification of (1) by violating the boundary condition at the points of \mathcal{E} . As consequence, a Fredholm problem arises, but the restriction $u|_{\partial\Omega}$ has a finite jump at \mathcal{E} . What is the common feature of the degenerate problems, independently of the type of ℓ , is that the solution “loses regularity” near the set of tangency from the data of (1) in contrast to the non-degenerate case when any solution gains two derivatives from f and one derivative from φ . Roughly speaking, that loss of smoothness depends on the *order of contact* between ℓ and $\partial\Omega$ and is given by the *subelliptic* estimates obtained for the solutions of degenerate problems (cf. [4, 5, 6, 11]). Precisely, if ℓ has a contact of order k with $\partial\Omega$ then the solution of (1) gains $2 - k/(k + 1)$ derivatives from f and $1 - k/(k + 1)$ derivatives from φ .

(a) neutral vector field ℓ (b) emergent vector field ℓ (c) submergent vector field ℓ

For what concerns the geometric structure of \mathcal{E} , it was supposed initially to be a submanifold of $\partial\Omega$ of codimension one. Melin and Sjöstrand [12] and Paneah [17] were the first to study the Poincaré problem (1) in a more general situation when \mathcal{E} is a massive subset of $\partial\Omega$ with positive surface measure, allowing \mathcal{E} to contain arcs of ℓ -trajectories of *finite* length. Their results were extended by Winzell ([23, 24]) to the framework of Hölder’s spaces who studied (1) assuming $C^{1,\alpha}$ -smoothness of the coefficients of \mathcal{L} . It is worth noting that ℓ has automatically an *infinite* order of contact with $\partial\Omega$ when \mathcal{E} is a massive subset of the boundary.

To deal with non-linear Poincaré problems, however, we have to dispose of precise information on the linear problem (1) with coefficients less regular than C^∞ (see [13, 20, 21, 22]). Indeed, *a priori* estimates in $W^{2,p}$ for solutions to (1) would imply easily pointwise estimates for u and Du for suitable values of $p > 1$ through the Sobolev imbeddings. This way, we are naturally led to consider the problem (1) in a *strong* sense,

that is, to searching for solutions lying in $W^{2,p}$ which satisfy $\mathcal{L}u = f$ almost everywhere (a.e.) in Ω and $\mathcal{B}u = \varphi$ holds in the sense of trace on $\partial\Omega$.

In the papers [4, 5] by Guan and Sawyer solvability and precise subelliptic estimates have been obtained for (1) in $H^{s,p}$ -spaces ($\equiv W^{s,p}$ for integer s !). However, [4] treats operators with C^∞ -coefficients and this determines the technique involved and the results obtained, while in [5] the coefficients are $C^{0,\alpha}$ -smooth, but the field ℓ is of finite type, that is, it has a *finite* order of contact with $\partial\Omega$.

The main goal of this lecture is to derive *a priori* estimates in Sobolev's classes $W^{2,p}(\Omega)$ with *any* $p \in (1, \infty)$ for the solutions of the Poincaré problem (1), weakening both Winzell's assumptions on $C^{1,\alpha}$ -regularity of the coefficients of \mathcal{L} and these of Guan and Sawyer on the *finite type* of ℓ . We are dealing with the simpler case when γ preserves its sign on $\partial\Omega$ which means the field ℓ is of *neutral type*. Of course, the loss of smoothness mentioned, imposes some more regularity of the data near the set \mathcal{E} . We assume the coefficients of \mathcal{L} to be Lipschitz continuous near \mathcal{E} while only continuity (and even discontinuity controlled in VMO) is allowed away from \mathcal{E} . Similarly, ℓ is a Lipschitz vector field on $\partial\Omega$ with Lipschitz continuous first derivatives near \mathcal{E} , and *no restrictions* on the order of contact with $\partial\Omega$ are required. Regarding the tangency set \mathcal{E} , it may have positive surface measure and is restricted only to a sort of *non-trapping* condition that all trajectories of ℓ through the points of \mathcal{E} are non-closed and leave \mathcal{E} in a finite time.

The technique adopted is based on a dynamical system approach employing the fact that $\partial u / \partial \ell$ is a local strong solution, near \mathcal{E} , to a Dirichlet-type problem with right-hand side depending on the solution u itself. Application of the L^p -estimates for such problems leads to the functional inequality (26) for suitable $W^{2,p}$ -norms of u on a family of subdomains which, starting away from \mathcal{E} , evolve along the ℓ -trajectories and exhaust a sort of their tubular neighbourhoods. Fortunately, that is an inequality with advanced argument and the desired $W^{2,p}$ -estimate follows by iteration with respect to the curvilinear parameter on the trajectories of ℓ . Another advantage of this approach is the *improving-of-integrability* property obtained for the solutions of (1). Roughly speaking, it asserts that the problem (1), even if a *degenerate* one, behaves as an *elliptic* problem for what concerns the degree p of integrability. In other words, the second derivatives of any solution to (1) will have the same rate of integrability as f and φ . We refer the reader to the paper [16] for outgrowths of the $W^{2,p}$ -a priori estimates, such as uniqueness in $W^{2,p}(\Omega)$, $\forall p > 1$, of the strong solutions to (1) as well as its Fredholmness.

Concluding this introduction, we should mention the articles [8, 9, 15] where similar results have been obtained by different technique in the particular case when the tangency set \mathcal{E} contains trajectories of ℓ with positive, but *small enough* lengths.

1. HYPOTHESES AND THE MAIN RESULT

Hereafter $\Omega \subset \mathbb{R}^n$, $n \geq 3$, will be a bounded domain with reasonably smooth boundary and $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$ stands for the unit *outward* normal to $\partial\Omega$ at $x \in \partial\Omega$. Consider a unit vector field $\ell(x) = (\ell^1(x), \dots, \ell^n(x))$ on $\partial\Omega$ and let $\ell(x) = \tau(x) + \gamma(x)\nu(x)$, where $\tau: \partial\Omega \rightarrow \mathbb{R}^n$ is the projection of $\ell(x)$ on the hyperplane tangent to $\partial\Omega$ at $x \in \partial\Omega$ and $\gamma: \partial\Omega \rightarrow \mathbb{R}$ is the inner product $\gamma(x) := \ell(x) \cdot \nu(x)$. The set of zeroes of γ ,

$$\mathcal{E} := \{x \in \partial\Omega: \gamma(x) = 0\},$$

is indeed the subset of $\partial\Omega$ where the field $\ell(x)$ becomes tangent to it.

Fix $\mathcal{N} \subset \overline{\Omega}$ to be a closed neighbourhood of \mathcal{E} in $\overline{\Omega}$. We suppose \mathcal{L} is a uniformly elliptic operator with measurable coefficients, satisfying

$$(2) \quad \lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{a.a. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad a^{ij}(x) = a^{ji}(x)$$

for some positive constant λ . Regarding the regularity of the data, we assume

$$(3) \quad \begin{cases} a^{ij} \in VMO(\Omega) \cap C^{0,1}(\mathcal{N}), \\ \partial\Omega \in C^{1,1}, \quad \partial\Omega \cap \mathcal{N} \in C^{2,1}, \quad \ell^i \in C^{0,1}(\partial\Omega) \cap C^{1,1}(\partial\Omega \cap \mathcal{N}) \end{cases}$$

with $VMO(\Omega)$ being the Sarason class of functions of vanishing mean oscillation and $C^{k,1}$ denotes the space of functions with Lipschitz continuous k -th order derivatives. Let us point out that (2), (3) and the Rademacher theorem give $a^{ij} \in L^\infty(\Omega) \cap W^{1,\infty}(\mathcal{N})$. For what concerns the boundary operator \mathcal{B} , we assume

$$(4) \quad \begin{cases} \gamma(x) = \ell(x) \cdot \nu(x) \geq 0 \quad \forall x \in \partial\Omega, \quad \text{and} \\ \text{the arcs of the } \ell\text{-trajectories lying in } \mathcal{E} \text{ (which coincide with these of } \tau) \\ \text{are all non-closed and of finite lengths.} \end{cases}$$

The first assumption simply means that $\ell(x)$ is either tangential to $\partial\Omega$ or is directed outwards Ω , that is, the field ℓ is of *neutral type* on $\partial\Omega$, while the second one is a sort of *non-trapping* condition on the tangency set \mathcal{E} . It implies that the ℓ -integral curves *leave* \mathcal{E} in a *finite time* in both directions.

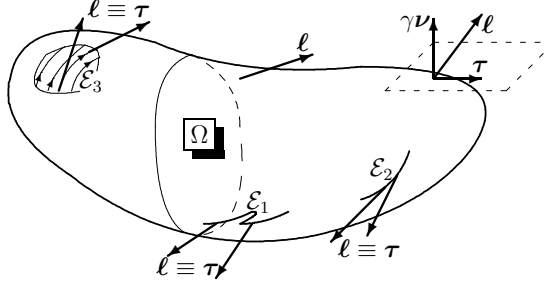


FIGURE 1. The set of tangency \mathcal{E} is the union $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where $\text{codim}_{\partial\Omega}\mathcal{E}_1 = \text{codim}_{\partial\Omega}\mathcal{E}_2 = 1$ while $\text{meas}_{\partial\Omega}\mathcal{E}_3 > 0$. The vector field ℓ is transversal to \mathcal{E}_1 and tangent to \mathcal{E}_2 . Actually, \mathcal{E}_2 consists of an arc of τ -trajectory, whereas \mathcal{E}_3 is union of such arcs.

Throughout the text $W^{k,p}$ stands for the Sobolev class of functions with L^p -summable weak derivatives up to order $k \in \mathbb{N}$ while $W^{s,p}(\partial\Omega)$ with $s > 0$ non-integer and $p \in (1, +\infty)$, is the Sobolev space of fractional order on $\partial\Omega$. Further, we use the standard parameterization $t \mapsto \psi_{\mathbf{L}}(t; x)$ for the *trajectory* (equivalently, *phase curve*, *maximal integral curve*) of a given vector field \mathbf{L} passing through a point x , that is, $\partial_t \psi_{\mathbf{L}}(t; x) = \mathbf{L}(\psi_{\mathbf{L}}(t; x))$ and $\psi_{\mathbf{L}}(0; x) = x$.

We will employ below an extension of the field ℓ near $\partial\Omega$ which preserves therein its regularity and geometric properties. All the results and proofs in the sequel work for such an *arbitrary* ℓ -extension but, in order to make more evident some geometric constructions, we prefer to introduce a *special* extension as follows. For each $x \in \mathbb{R}^n$ near $\partial\Omega$ set $d(x) = \text{dist}(x, \partial\Omega)$ and define $\Gamma := \{x \in \mathbb{R}^n : d(x) \leq d_0\}$ with small $d_0 > 0$. Letting $\Omega_0 := \Omega \setminus \Gamma$ and $y(x) \in \partial\Omega$ for the unique point closest to $x \in \Gamma$, we have (see [3, Chapter 14]) $y(x) \in C^{0,1}(\Gamma)$ while $y(x) \in C^{1,1}$ near \mathcal{E} . Regarding the distance function $d(x) = |x - y(x)|$, it is Lipschitz continuous in Γ and inherits the regularity of $\partial\Omega$ at

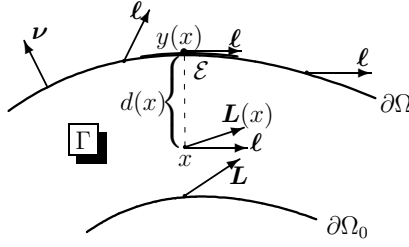


FIGURE 2.

$y(x)$ when considered on the parts of Γ lying in/out Ω , but its normal derivative has a finite jump on $\partial\Omega$. Anyway, it is a routine to check $(d(x))^2 \in C^{1,1}(\Gamma)$. Setting $\mathbf{L}(x)$ for the normalized representative of $\ell(y(x)) + (d(x))^2 \nu(y(x)) \forall x \in \Gamma$, it results $|\mathbf{L}(x)| = 1$, $\mathbf{L}|_{\partial\Omega} = \ell$, $\mathbf{L}|_{\mathcal{E}} = \tau$ and $\mathbf{L} \in C^{0,1}(\Gamma) \cap C^{1,1}(\Gamma \cap \mathcal{N})$. Moreover, the field \mathbf{L} is strictly transversal to $\partial\Omega_0$.

As consequence of the non-trapping condition (4), the compactness of \mathcal{E} and the semi-continuity properties of the lengths of the τ -maximal integral curves, it is not hard to get that (see [24, Proposition 3.1] and [22, Proposition 3.2.4]) *under the hypotheses (3) and (4), there is a finite upper bound κ_0 for the arclengths of the τ -trajectories lying in \mathcal{E} . Moreover, each point of Γ can be reached from $\partial\Omega_0$ by an \mathbf{L} -integral curve of length at most $\kappa = \text{const} > 0$.*

In what follows, the letter C will denote a generic constant depending on known quantities defined by the data of (1), that is, on n, p, λ , the respective norms of the coefficients of \mathcal{L} and \mathcal{B} in Ω and \mathcal{N} , the regularity of $\partial\Omega$ and the constants κ_0 and κ .

In order to control precisely the regularity of u near the tangency set \mathcal{E} , we have to introduce the appropriate functional spaces. For, take an arbitrary $p \in (1, \infty)$ and define the Banach spaces

$$\mathcal{F}^p(\Omega, \mathcal{N}) := \{f \in L^p(\Omega) : \partial f / \partial \mathbf{L} \in L^p(\mathcal{N})\}$$

equipped with norm $\|f\|_{\mathcal{F}^p(\Omega, \mathcal{N})} := \|f\|_{L^p(\Omega)} + \|\partial f / \partial \mathbf{L}\|_{L^p(\mathcal{N})}$, and

$$\Phi^p(\partial\Omega, \mathcal{N}) := \left\{ \varphi \in W^{1-1/p, p}(\partial\Omega) : \varphi \in W^{2-1/p, p}(\partial\Omega \cap \mathcal{N}) \right\}$$

normed by $\|\varphi\|_{\Phi^p(\partial\Omega, \mathcal{N})} := \|\varphi\|_{W^{1-1/p, p}(\partial\Omega)} + \|\varphi\|_{W^{2-1/p, p}(\partial\Omega \cap \mathcal{N})}$.

Our main result asserts that the couple $(\mathcal{L}, \mathcal{B})$ *improves the integrability* of solutions to (1) for any p in the range $(1, \infty)$ and, moreover, provides for an *a priori estimate* in the L^p -Sobolev scales for any such solution.

Theorem 1. *Under the hypotheses (2)–(4) let $u \in W^{2,p}(\Omega)$ be a strong solution of the problem (1) with $f \in \mathcal{F}^q(\Omega, \mathcal{N})$ and $\varphi \in \Phi^q(\partial\Omega, \mathcal{N})$ where $1 < p \leq q < \infty$.*

Then $u \in W^{2,q}(\Omega)$ and there is an absolute constant C such that

$$(5) \quad \|u\|_{W^{2,q}(\Omega)} \leq C \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})} \right).$$

Let us point out reader's attention that the directional derivative $\partial u / \partial \mathbf{L}$ of each $W^{2,p}$ -solution to (1) belongs to $W^{2,p}(\mathcal{N})$. For, $\partial u / \partial \mathbf{L} \in W^{1,p}(\mathcal{N})$ and taking the difference quotients in (1) in the direction of \mathbf{L} (cf. [3, Chapter 8 and Lemma 7.24]) gives that

$\partial u / \partial \mathbf{L} \in W^{2,p}(\mathcal{N})$ is a strong local solution to the Dirichlet problem

$$(6) \quad \begin{cases} \mathcal{L} \left(\frac{\partial u}{\partial \mathbf{L}} \right) = \frac{\partial f}{\partial \mathbf{L}} + 2a^{ij} D_j L^k D_{ki} u + a^{ij} D_{ij} L^k D_k u - \frac{\partial a^{ij}}{\partial \mathbf{L}} D_{ij} u & \text{a.e. } \mathcal{N}, \\ \frac{\partial u}{\partial \mathbf{L}} = \varphi & \text{on } \partial \Omega \cap \mathcal{N} \end{cases}$$

where $\mathbf{L}(x) = (L^1(x), \dots, L^n(x)) \in C^{1,1}(\mathcal{N})$. Therefore, once having proved $u \in W^{2,q}(\Omega)$ and the estimate (5), we have

$$\|\partial u / \partial \mathbf{L}\|_{W^{2,q}(\tilde{\mathcal{N}})} \leq C' (\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial \Omega, \mathcal{N})})$$

for any *closed* neighbourhood $\tilde{\mathcal{N}}$ of \mathcal{E} in $\overline{\Omega}$, $\tilde{\mathcal{N}} \subset \mathcal{N}$, by means of the L^p -theory of uniformly elliptic equations (see [1] or [3, Chapter 9]). In other words, *if a strong solution u to (1) belongs to $W^{2,q}(\Omega)$ then $\partial u / \partial \mathbf{L} \in W^{2,q}(\mathcal{N})$ automatically, provided $f \in \mathcal{F}^q(\Omega, \mathcal{N})$ and $\varphi \in \Phi^q(\partial \Omega, \mathcal{N})$.*

2. PROOF OF THEOREM 1

Fix hereafter $\mathcal{N}' \subset \mathcal{N}'' \subset \mathcal{N}$ to be closed neighbourhoods of \mathcal{E} in $\overline{\Omega}$ with \mathcal{N}'' so “narrow” that $\mathcal{N}'' \subset \Omega \setminus \Omega_0$ (see Figure 3). The next result is an immediate consequence of $\gamma(x) > 0 \ \forall x \in \partial \Omega \setminus \mathcal{N}'$ and the L^p -theory of *regular* oblique derivative problems for uniformly elliptic operators with *VMO* principal coefficients (cf. [7, Theorem 2.3.1]).

Proposition 2. *Assume (2), (3) and $\gamma(x) > 0 \ \forall x \in \Omega \setminus \mathcal{E}$, and let $u \in W^{2,p}(\Omega)$ be a solution to (1) with $f \in L^q(\Omega)$ and $\varphi \in W^{1-1/q,q}(\partial \Omega)$, where $1 < p \leq q < \infty$.*

Then $u \in W^{2,q}(\Omega \setminus \mathcal{N}')$ and there is a constant such that

$$(7) \quad \|u\|_{W^{2,q}(\Omega \setminus \mathcal{N}')} \leq C (\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\varphi\|_{W^{1-1/q,q}(\partial \Omega)}).$$

To derive the improving-of-integrability near the tangency set \mathcal{E} , we consider any solution of the problem (1) for which $a^{ij}, \partial a^{ij} / \partial \mathbf{L} \in L^\infty(\mathcal{N})$ in view of (3)¹ and $f, \partial f / \partial \mathbf{L} \in L^q(\mathcal{N})$ and $\varphi \in W^{2-1/q,q}(\partial \Omega \cap \mathcal{N})$ by hypotheses.

Lemma 3. *Under the assumptions of Theorem 1, the solution u of (1) belongs to $u \in W^{2,q}(\mathcal{N}'')$ and there is a constant such that*

$$(8) \quad \|u\|_{W^{2,q}(\mathcal{N}'')} \leq C (\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial \Omega, \mathcal{N})}).$$

Proof. Take an arbitrary point $x_0 \in \mathcal{E}$. According to (4), the \mathbf{L} -trajectory through x_0 leaves \mathcal{E} in both directions for a finite time, that is, $\psi_{\mathbf{L}}(t^-; x_0) \in \mathcal{N}'' \setminus \mathcal{N}'$, $\psi_{\mathbf{L}}(t^+; x_0) \in \mathbb{R}^n \setminus \overline{\Omega}$ (see Figure 3) for suitable $t^- < 0 < t^+$.

Set \mathcal{H} for the $(n-1)$ -dimensional hyperplane through x_0 and orthogonal to $\mathbf{L}(x_0)$, and define

$$B_r(x_0) := \{x \in \mathcal{H}: |x - x_0| < r\}$$

with $r > 0$ to be chosen later. It follows from the Picard inequality² that if r is small enough, then the flow of $B_r(x_0)$ along the \mathbf{L} -trajectories at time t^- ,

$$B'_r(x_0) := \psi_{\mathbf{L}}(t^-; B_r(x_0)) := \{\psi_{\mathbf{L}}(t^-; y): y \in B_r(x_0)\}$$

is *entirely contained* in $\mathcal{N}'' \setminus \mathcal{N}'$ whence $B'_r(x_0) \cap \mathcal{E} = \emptyset$. The set

$$\Theta_r := \{\psi_{\mathbf{L}}(t; x'): x' \in B'_r(x_0), \quad t \in (0, t^+ - t^-)\}$$

¹It will be clear from the considerations given below that instead of Lipschitz continuity of the coefficients of \mathcal{L} in \mathcal{N} as (3) asks, it suffices to have essentially bounded their directional derivatives with respect to the field \mathbf{L} .

² $|\psi_{\mathbf{L}}(t; x') - \psi_{\mathbf{L}}(t; x'')| \leq e^{t\|\mathbf{L}\|_{C^1(\mathcal{N})}} |x' - x''|$ for all $x', x'' \in \mathcal{N}$.

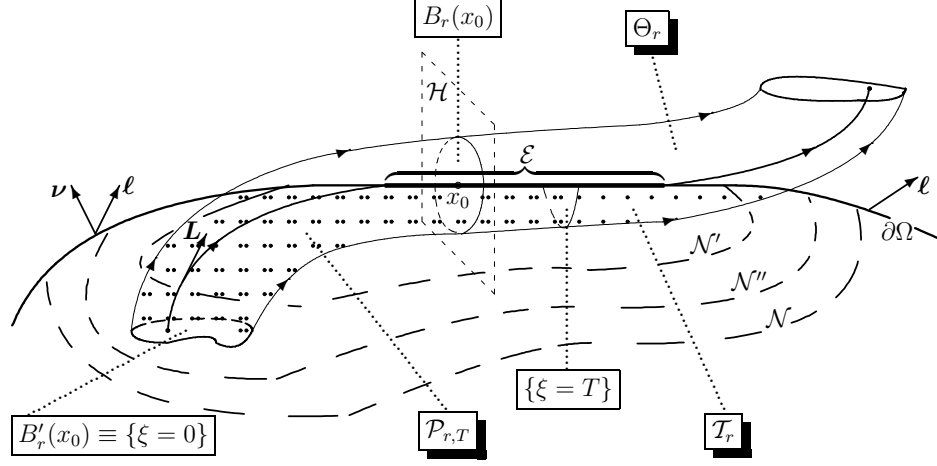


FIGURE 3. \mathcal{T}_r is the dotted set, while the double-dotted one is $\mathcal{P}_{r,T}$.

is an n -dimensional neighbourhood of the \mathbf{L} -trajectory through x_0 and defining

$$\mathcal{T}_r := \Theta_r \cap \Omega,$$

the boundary $\partial\mathcal{T}_r$ is composed of the “base” $B'_r(x_0)$ and the “lateral” components $\partial_1\mathcal{T}_r := \partial\mathcal{T}_r \cap \partial\Omega$ and $\partial_2\mathcal{T}_r := (\partial\mathcal{T}_r \cap \Omega) \setminus B'_r(x_0)$. Indeed, $\mathcal{T}_r \subset \mathcal{N}''$ if $r > 0$ is small enough.

We will derive (8) in \mathcal{T}_r after that the desired estimate will follow by covering the compact $\mathcal{E} \subset \partial\Omega$ by a finite number of sets like $\overline{\mathcal{T}_r}$. Our strategy is based on a representation of $u(x)$ in \mathcal{T}_r by means of $u(x')$ with $x' = \psi_{\mathbf{L}}(-\xi(x); x) \in B'_r(x_0)$ for some $\xi(x) > 0$, and the integral of $\partial u / \partial \mathbf{L}$ along the \mathbf{L} -trajectory joining x' with x . Thus the Sobolev norm of u will be expressed by the respective norm of $\partial u / \partial \mathbf{L}$ and that of u itself near $B'_r(x_0)$ where we dispose of (7). Concerning $\partial u / \partial \mathbf{L}$, it is a local solution of Dirichlet problem near \mathcal{E} with right-hand side depending on u .

Let $\mu: \mathcal{H} \rightarrow \mathbb{R}^+$ be a C^∞ cut-off function such that

$$(9) \quad \mu(y) = \begin{cases} 1 & y \in B_{r/2}(x_0), \\ 0 & y \in \mathcal{H} \setminus B_{3r/4}(x_0) \end{cases}$$

and extend it to \mathbb{R}^n as constant on the \mathbf{L} -trajectory through $y \in \mathcal{H}$. The function $U(x) := \mu(x)u(x)$ is a $W^{2,p}(\mathcal{N})$ -solution of

$$(10) \quad \begin{cases} \mathcal{L}U = F(x) := \mu f + 2a^{ij}D_j\mu D_i u + ua^{ij}D_{ij}\mu & \text{a.e. } \mathcal{T}_r, \\ \partial U / \partial \mathbf{L} = \Phi := \begin{cases} \mu\varphi & \text{on } \partial_1\mathcal{T}_r, \\ 0 & \text{near } \partial_2\mathcal{T}_r, \\ \mu\partial u / \partial \mathbf{L} & \text{on } B'_r(x_0) \subset \mathcal{N}'' \setminus \mathcal{N}'. \end{cases} \end{cases}$$

Indeed, $u \in W^{2,p}(\mathcal{N})$ implies $Du \in L^{np/(n-p)}$ if $p < n$ and $Du \in L^s \forall s > 1$ when $p \geq n$, whence $F \in L^{q'}(\mathcal{N})$ with

$$(11) \quad q' := \begin{cases} \min \left\{ q, \frac{np}{n-p} \right\} & \text{if } p < n, \\ q & \text{if } p \geq n. \end{cases}$$

Further, $\partial F/\partial \mathbf{L} \in L^{q'}(\mathcal{N}'')$ as consequence of (6), $\partial u/\partial \mathbf{L} \in W^{2,q}(\mathcal{N}'' \setminus \mathcal{N}')$ by Proposition 2 whence $\Phi \in W^{2-1/q,q}(\partial \mathcal{T}_r)$. Thus (2), (3), $\mathcal{T}_r \subset \mathcal{N}''$ and (6) give that

$$V(x) := \partial U/\partial \mathbf{L}$$

is a $W^{2,p}(\mathcal{T}_r)$ -solution of the Dirichlet problem

$$(12) \quad \begin{cases} \mathcal{L}V = \partial F/\partial \mathbf{L} + 2a^{ij}D_j L^k D_{ik}U + a^{ij}D_{ij}L^k D_k U - \frac{\partial a^{ij}}{\partial \mathbf{L}} D_{ij}U & \text{a.e. } \mathcal{T}_r, \\ V = \Phi & \text{on } \partial \mathcal{T}_r. \end{cases}$$

Now we pass from $x \in \Theta_r$ into the new variables (x', ξ) with $x' = \psi_{\mathbf{L}}(-\xi(x); x) \in B'_r(x_0)$ and $\xi: \Theta_r \rightarrow (0, t^+ - t^-)$, $\xi(x) \in C^{1,1}(\Theta_r)$. The transform $x \mapsto (x', \xi)$ defines a $C^{1,1}$ -diffeomorphism because the field \mathbf{L} is transversal to $B'_r(x_0)$. Moreover, $\partial/\partial \mathbf{L} \equiv \partial/\partial \xi$, $\psi_{\mathbf{L}}(t; x') = (x', t)$ and $V(x', \xi) = \partial U(x', \xi)/\partial \xi$ as $(x', \xi) \in \mathcal{T}_r$. Since $V(x', \xi)$ is an absolutely continuous function in ξ for a.a. $x' \in B'_r(x_0)$ (after redefining it, if necessary, on a set of zero measure) we get

$$(13) \quad U(x', \xi) = U(x', 0) + \int_0^\xi V(x', t) dt \quad \text{for a.a. } (x', \xi) \in \mathcal{T}_r,$$

where the point $(x', 0) \in B'_r(x_0)$ lies in $\mathcal{N}'' \setminus \mathcal{N}'$ and $U(x', 0) \in W^{2,q}$ there by Proposition 2, the Fubini theorem and [14, Remark 2.1]. Passing to the new variables (x', ξ) in (12), taking the derivatives of (13) up to second order and substituting them into the right-hand side of (12), this last reads

$$(14) \quad \begin{cases} \mathcal{L}'V = F_1(x', \xi) + \int_0^\xi \mathcal{D}_2(\xi)V(x', t) dt & \text{a.e. } \mathcal{T}_r, \\ V = \Phi & \text{on } \partial \mathcal{T}_r, \end{cases}$$

where \mathcal{L}' is the operator \mathcal{L} in terms of $(x', \xi) = (x'_1, \dots, x'_{n-1}, \xi)$,

$$(15) \quad \begin{aligned} F_1(x', \xi) &:= \partial F/\partial \mathbf{L} + \mathcal{D}_1 V(x', \xi) + \mathcal{D}'_1 U(x', \xi) + \mathcal{D}'_2 U(x', 0), \\ \mathcal{D}_2(\xi)V(x', t) &:= \sum_{i,j=1}^{n-1} A^{ij}(x', \xi) D_{x'_i x'_j} V(x', t), \quad A^{ij} \in L^\infty, \end{aligned}$$

\mathcal{D}_1 , \mathcal{D}'_1 , \mathcal{D}'_2 are linear differential operators with L^∞ -coefficients, $\text{ord } \mathcal{D}_1 = \text{ord } \mathcal{D}'_1 = 1$, $\text{ord } \mathcal{D}'_2 = 2$. The Sobolev imbedding theorem implies $F_1 \in L^{q'}(\mathcal{T}_r)$ with q' given by (11) as consequence of $\partial F/\partial \mathbf{L} \in L^{q'}(\mathcal{N}'')$, $U(x', 0) \in W^{2,q}(B'_r(x_0))$ and $U, V \in W^{2,p}(\mathcal{N}'')$. Nevertheless the second-order operator $\mathcal{D}_2(\xi)$ has a quite rough characteristic form which is neither symmetric nor sign-definite, the improving-of-integrability holds for (14) thanks to the particular structure of \mathcal{T}_r as union of \mathbf{L} -trajectories through $B'_r(x_0)$. Actually, we will show that if $V \in W^{2,q'}$ on a subset of \mathcal{T}_r with $\xi < T$, then V remains a $W^{2,q'}$ -function on a larger subset with $\xi < T + r$ for small enough r , after that the higher integrability of U will follow from Proposition 2 and (13). For, take an arbitrary $T \in (0, t^+ - t^-)$ and define

$$\mathcal{P}_{r,T} := \{(x', \xi) \in \mathcal{T}_r : \xi < T\}.$$

For a fixed $r > 0$, $\{\mathcal{P}_{r,T}\}_{T \geq 0}$ is a non-decreasing family of domains exhausting \mathcal{T}_r and $\mathcal{P}_{r,T} \equiv \mathcal{T}_r$ for values of T greater than the *maximal exit-time*

$$T_{\max} := \sup_{x' \in B'_r(x_0)} \sup \{t > 0 : \psi_{\mathbf{L}}(t; x') \in \Omega, x' \in B'_r(x_0)\}.$$

Proposition 4. *Let $T \in (0, t^+ - t^-)$ and consider the solution $V \in W^{2,p}(\mathcal{T}_r)$ of the problem (14). Suppose $V \in W^{2,q'}(\mathcal{P}_{r,T})$ where q' is given by (11).*

There exists an $r_0 > 0$ such that $V \in W^{2,q'}(\mathcal{P}_{r,T+r})$ for all $r < r_0$.

Proof. There are three possible cases to be distinguished.

Case A: $T + 3r < T_{\max}$. We have $\mathcal{P}_{r,T} \subset \mathcal{P}_{r,T+3r} \subset \mathcal{T}_r \equiv \mathcal{P}_{r,T_{\max}}$ and consider the C^∞ -function $\eta: \mathbb{R} \rightarrow [0, 1]$ such that

$$(16) \quad \eta(\xi) = \begin{cases} 1 & \text{as } \xi \in (-\infty, T+r], \\ \text{strictly decreases} & \text{as } \xi \in (T+r, T+2r), \\ 0 & \text{as } \xi \geq T+2r. \end{cases}$$

Setting $\tilde{V}(x', \xi) := \eta(\xi)V(x', \xi)$, it follows $\mathcal{L}'\tilde{V} = \eta(\mathcal{L}'V) + \mathcal{L}_1V$ where \mathcal{L}_1 is a first-order differential operator with L^∞ -coefficients depending on these of \mathcal{L}' and on the derivatives of η . Therefore,

$$(17) \quad \begin{aligned} \mathcal{L}'\tilde{V} &= \eta F_1 + \mathcal{L}_1V + \eta(\xi) \int_0^\xi \mathcal{D}_2(\xi)V(x', t)dt \\ &= \eta F_1 + \mathcal{L}_1V + \int_0^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt \end{aligned}$$

because $\mathcal{D}_2(\xi)$ is a second-order operator acting in the x' -variables only.

We set $\Omega_r \subset \mathcal{P}_{r,T+3r} \setminus \mathcal{P}_{r,T-3r}$ for a $C^{1,1}$ -smooth domain containing $\mathcal{P}_{3r/4,T+2r} \setminus \mathcal{P}_{3r/4,T-2r}$ and such that

$$r^{-1}\Omega_r := \left\{ (\tilde{y}', \tilde{\xi}): \tilde{y}' = x'/r, \tilde{\xi} = (\xi - T)/r, (x', \xi) \in \Omega_r \right\} \in C^{1,1}$$

uniformly in r . The boundary $\partial\Omega_r$ consists of the “lateral” parts $\partial_1\Omega_r := \partial\Omega_r \cap \partial\Omega$ and $\partial_2\Omega_r := \partial\Omega_r \cap \Omega \cap \{\xi \in (T-2r, T+2r)\} \subset (\mathcal{P}_{r,T+2r} \setminus \mathcal{P}_{r,T-2r}) \setminus (\mathcal{P}_{3r/4,T+2r} \setminus \mathcal{P}_{3r/4,T-2r})$, and of two $C^{1,1}$ -smooth components $\partial\Omega_r^\pm$ lying in $\mathcal{P}_{r,T+3r} \setminus \mathcal{P}_{r,T+2r}$ and $\mathcal{P}_{r,T-2r} \setminus \mathcal{P}_{r,T-3r}$, respectively. The properties of μ (cf. (9)) ensure $U \equiv 0$, $V \equiv 0$, $\tilde{V} \equiv 0$ on $\mathcal{T}_r \setminus \mathcal{T}_{3r/4}$ whence $\tilde{V} \equiv 0$ near $\partial_2\Omega_r$.

For an arbitrary $(x', \xi) \in \Omega_r$, the factor $\eta(\xi)/\eta(t)$ in (17) vanishes when $\xi \geq T+2r$ while $\eta(\xi)/\eta(t) \leq 1$ because η decreases in $(T+r, T+2r)$. Moreover, $|\xi - T| < 3r$ for $(x', \xi) \in \Omega_r$ and

$$\begin{aligned} \int_0^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt &= \int_0^T \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt + \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt \\ &= \eta(\xi) \int_0^T \mathcal{D}_2(\xi)V(x', t)dt + \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt \end{aligned}$$

by means of (15) and since $\eta(t) = \eta(T) = 1$ as $t \leq T$.

We get from (14) and (17) that $\tilde{V} \in W^{2,p}(\Omega_r)$ solves the Dirichlet problem

$$(18) \quad \begin{cases} \mathcal{L}'\tilde{V} = F_2(x', \xi) + \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi)\tilde{V}(x', t)dt & \text{a.a. } (x', \xi) \in \Omega_r, \\ \tilde{V} = \tilde{\Phi} := \eta\Phi = \begin{cases} \eta\mu\varphi \in W^{2-1/q,q} & \text{on } \partial_1\Omega_r \quad (\text{by (10)}), \\ 0 & \text{on } \partial_2\Omega_r \quad (\text{by (10)}), \\ 0 & \text{on } \partial\Omega_r^+ \quad (\text{by (16)}), \\ V \in W^{2-1/q',q'} & \text{on } \partial\Omega_r^- \quad (\text{since } \xi < T-2r \text{ and } V \in W^{2,q'}(\mathcal{P}_{r,T})) \end{cases} \end{cases}$$

where, recalling $V \in W^{2,q'}(\mathcal{P}_{r,T})$, we have

$$(19) \quad F_2(x', \xi) := \eta F_1 + \mathcal{L}_1V + \eta(\xi) \int_0^T \mathcal{D}_2(\xi)V(x', t)dt \in L^{q'}(\Omega_r).$$

We are going to prove now that $\tilde{V} \in W^{2,q'}(\Omega_r)$ for small enough $r > 0$, whence it will follow $V \in W^{2,q'}(\mathcal{P}_{r,T+r})$ in view of (16) and $V \equiv 0$ near $\partial_2\Omega_r$. The claim is obvious if $q' = p$ because $V \in W^{2,p}(\mathcal{T}_r)$. Otherwise, take an arbitrary $s \in [p, q']$ and denote by $W_*^{2,s}(\Omega_r)$ the Sobolev space $W^{2,s}(\Omega_r)$ normed with

$$\|u\|_{W_*^{2,s}(\Omega_r)} := \|u\|_{L^s(\Omega_r)} + r\|Du\|_{L^s(\Omega_r)} + r^2\|D^2u\|_{L^s(\Omega_r)}.$$

Define now the operator $\mathfrak{F}: W_*^{2,s}(\Omega_r) \rightarrow W_*^{2,s}(\Omega_r)$ as follows: for any $w \in W_*^{2,s}(\Omega_r)$ the image $\mathfrak{F}w \in W_*^{2,s}(\Omega_r)$ is the *unique* solution of the Dirichlet problem

$$(20) \quad \begin{cases} \mathcal{L}'(\mathfrak{F}w) = F_2 + \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi) w(x', t) dt \in L^s(\Omega_r) & \text{a.a. } (x', \xi) \in \Omega_r, \\ \mathfrak{F}w = \tilde{\Phi} \in W^{2-1/s,s}(\partial\Omega_r) & \text{on } \partial\Omega_r. \end{cases}$$

We will prove that \mathfrak{F} is a contraction for small values of r . For this goal, take arbitrary $w_1, w_2 \in W_*^{2,s}(\Omega_r)$. The difference $\mathfrak{F}w_1 - \mathfrak{F}w_2$ solves

$$(21) \quad \begin{cases} \mathcal{L}'(\mathfrak{F}w_1 - \mathfrak{F}w_2) = \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi) (w_1 - w_2)(x', t) dt & \text{a.a. } (x', \xi) \in \Omega_r, \\ \mathfrak{F}w_1 - \mathfrak{F}w_2 = 0 & \text{on } \partial\Omega_r. \end{cases}$$

In order to apply the L^s -*a priori* estimates from [1] or [3] for the solutions of (21), we have to control the dependence on r therein. For, we recall that $r^{-1}\Omega_r \in C^{1,1}$ uniformly in r and apply a standard approach consisting of dilation of Ω_r onto $r^{-1}\Omega_r$, reduction of the problem (21) to a new one in variables $(\tilde{y}', \tilde{\xi}) \in r^{-1}\Omega_r$, application of the L^s -estimates from [3, Theorem 9.17] and finally turning back to (21) (see the Proof of Lemma 2.2, Eq. (2.12) in [14]). This way, one gets

$$(22) \quad \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{W_*^{2,s}(\Omega_r)} \leq Cr^2 \left\| \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi) (w_1 - w_2)(x', t) dt \right\|_{L^s(\Omega_r)}$$

where the constant C is independent of r . Jensen's integral inequality yields

$$r^2 \left\| \int_T^\xi \frac{\eta(\xi)}{\eta(t)} \mathcal{D}_2(\xi) (w_1 - w_2)(x', t) dt \right\|_{L^s(\Omega_r)} \leq C \max_{(x', \xi) \in \Omega_r} |\xi - T| \|w_1 - w_2\|_{W_*^{2,s}(\Omega_r)}$$

and thus (22) rewrites into

$$\|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{W_*^{2,s}(\Omega_r)} \leq C \max_{(x', \xi) \in \Omega_r} |\xi - T| \|w_1 - w_2\|_{W_*^{2,s}(\Omega_r)}.$$

We have $\max_{(x', \xi) \in \Omega_r} |\xi - T| < 3r$, C is independent of r and therefore \mathfrak{F} will be really a contraction from $W_*^{2,s}(\Omega_r)$ into itself for any $s \in [p, q']$ if $r \leq r_0$ with r_0 under control and small enough. Fixing $r = r_0/2$, there is a unique fixed point of \mathfrak{F} in $W_*^{2,s}(\Omega_r)$ for all $s \in [p, q']$. However, $\tilde{V} \in W^{2,p}(\Omega_r)$ is already a fixed point of \mathfrak{F} since it solves (18) and therefore $\tilde{V} \in W^{2,q'}(\Omega_r)$. It follows $V \in W^{2,q'}(\mathcal{P}_{r,T+r})$ by means of $V \in W^{2,q'}(\mathcal{P}_{r,T})$, $\tilde{V} \equiv 0$ on $\mathcal{T}_r \setminus \mathcal{T}_{3r/4}$ and the properties of $\eta(\xi)$.

Case B: $T < T_{\max} \leq T + 3r$. We have $\mathcal{T}_r \setminus \mathcal{P}_{r,T} \neq \emptyset$, $\mathcal{P}_{r,T+3r} \equiv \mathcal{T}_r$ now and we do not need anymore the cut-off function η because $V = \partial U / \partial \mathbf{L} \equiv 0$ near the points of $\partial_2 \mathcal{T}_r$ where $\xi > T$ (cf. (9)). Thus, it suffices to repeat the above arguments with $\eta(\xi) \equiv 1 \forall \xi \in \mathbb{R}$ and $\Omega_r \in C^{1,1}$ defined as before when $\xi \leq T$ while $\mathcal{T}_{3r/4} \setminus \mathcal{P}_{3r/4,T} \subset (\Omega_r \cap \{\xi > T\}) \subset \mathcal{T}_r \setminus \mathcal{P}_{r,T}$

(cf. (9)). We have anyway a problem like (18) for $V \equiv \tilde{V}$ with boundary condition

$$V = \partial U / \partial \mathbf{L} = \begin{cases} \mu\varphi \in W^{2-1/q,q} & \text{on } \partial_1 \Omega_r = \partial \Omega_r \cap \partial \Omega, \\ 0 & \text{on } \partial_2 \Omega_r = \partial \Omega_r \cap \Omega \cap \{\xi > T - 3r\}, \\ V \in W^{2-1/q',q'} & \text{on } \partial \Omega_r^- \quad (\text{by hypothesis}). \end{cases}$$

Therefore, the procedure from *Case A* gives $V \in W^{2,q'}(\mathcal{P}_{r,T+3r})$.

Case C: $T_{\max} \leq T$. We have $\mathcal{P}_{r,T+r} \equiv \mathcal{P}_{r,T} \equiv \mathcal{T}_r$ now and thus the claim. \square

Proposition 5. *Suppose $r < r_0$ with r_0 given in Proposition 4. Then the solution V of the problem (14) lies in $W^{2,q}(\mathcal{T}_r)$ and satisfies the estimate*

$$(23) \quad \|V\|_{W^{2,q}(\mathcal{T}_r)} \leq C \left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial \Omega, \mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{T}_r)} \right).$$

Proof. We note that $V \in W^{2,q} \subseteq W^{2,q'}$ near $B'_r(x_0)$ in view of $B'_r(x_0) \subset \mathcal{N}'' \setminus \mathcal{N}'$, Proposition 2 and (6). Therefore, successive applications of Proposition 4 with increasing values of T will give $V \in W^{2,q'}(\mathcal{T}_r)$, $q' > p$. After that, in order to get $V \in W^{2,q}(\mathcal{T}_r)$, it suffices to put q' in the place of p in (11) and to repeat finitely many times the above arguments until $q' = q$.

To obtain (23), we take $T \in (0, t^+ - t^-)$ to be arbitrary, fix $r = r_0/2$, and consider the domains Ω_r defined in the proof of Proposition 4. Let $\tilde{V} = \eta V \in W^{2,q}(\mathcal{T}_r)$ solve (18) with η given by (16) in *Case A* and $\eta \equiv 1$ in *Case B*. Since \tilde{V} is a fixed point of the mapping $\mathfrak{F}: W^{2,q}(\Omega_r) \rightarrow W^{2,q}(\Omega_r)$, $\mathfrak{F}\tilde{V} = \tilde{V}$, we get

$$\|D^2 \tilde{V}\|_{L^q(\Omega_r)} = \|D^2(\mathfrak{F}\tilde{V})\|_{L^q(\Omega_r)} \leq \|D^2(\mathfrak{F}\tilde{V} - \mathfrak{F}0)\|_{L^q(\Omega_r)} + \|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)},$$

while

$$\|D^2(\mathfrak{F}w_1 - \mathfrak{F}w_2)\|_{L^q(\Omega_r)} \leq \theta \|D^2(w_1 - w_2)\|_{L^q(\Omega_r)} \quad \forall w_1, w_2 \in W^{2,q}(\Omega_r), \quad \theta < 1$$

because \mathfrak{F} is a contraction, (22) and the fact that $\mathcal{D}_2(\xi)$ is a homogeneous second-order operator (cf. (15)). This way, $\|D^2(\mathfrak{F}\tilde{V} - \mathfrak{F}0)\|_{L^q(\Omega_r)} \leq \theta \|D^2(\tilde{V} - 0)\|_{L^q(\Omega_r)} = \theta \|D^2 \tilde{V}\|_{L^q(\Omega_r)}$ and therefore

$$(24) \quad \|D^2 \tilde{V}\|_{L^q(\Omega_r)} \leq C \|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)}$$

with $\mathfrak{F}0 \in W^{2,q}(\Omega_r)$ being the unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}'(\mathfrak{F}0) = F_2 & \text{a.e. } \Omega_r, \\ \mathfrak{F}0 = \tilde{\Phi} & \text{on } \partial \Omega_r \end{cases}$$

(see (20)), for which the L^p -theory (cf. [3, Chapter 9]) gives

$$(25) \quad \|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)} \leq \|\mathfrak{F}0\|_{W^{2,q}(\Omega_r)} \leq C \left(\|F_2\|_{L^q(\Omega_r)} + \|\tilde{\Phi}\|_{W^{2-1/q,q}(\partial \Omega_r)} \right).$$

Direct applications, based on (19) and (15), yield

$$\begin{aligned}
\|F_2\|_{L^q(\Omega_r)} &= \left\| \eta F_1 + \mathcal{L}_1 V + \eta(\xi) \int_0^T \mathcal{D}_2(\xi) V(x', t) dt \right\|_{L^q(\Omega_r)} \\
&\leq C \left(\|\partial F / \partial \mathbf{L}\|_{L^q(\Omega_r)} + \|U\|_{W^{2,q}(\mathcal{N}'' \setminus \mathcal{N}')} + \|U\|_{W^{1,q}(\mathcal{T}_r)} + \|V\|_{W^{1,q}(\mathcal{T}_r)} \right. \\
&\quad \left. + \|D^2 V\|_{L^q(\mathcal{P}_{r,T})} \right) \\
&\leq C \left(\|\partial f / \partial \mathbf{L}\|_{L^q(\mathcal{N})} + \|u\|_{W^{2,q}(\mathcal{N}'' \setminus \mathcal{N}')} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{T}_r)} \right. \\
&\quad \left. + \|D^2 V\|_{L^q(\mathcal{P}_{r,T})} \right)
\end{aligned}$$

in view of (7), (10), $U = \mu u$, $V = \partial U / \partial \mathbf{L}$ and (9). Moreover,

$$\begin{aligned}
\|\tilde{\Phi}\|_{W^{2-1/q,q}(\partial\Omega_r)} &\leq C (\|\varphi\|_{W^{2-1/q,q}(\partial\Omega \cap \mathcal{N})} + \|V\|_{W^{2,q}(\mathcal{P}_{r,T})}) \\
&\leq C (\|\varphi\|_{W^{2-1/q,q}(\partial\Omega \cap \mathcal{N})} + \|V\|_{W^{1,q}(\mathcal{T}_r)} + \|D^2 V\|_{L^q(\mathcal{P}_{r,T})}) \\
&\leq C (\|\varphi\|_{W^{2-1/q,q}(\partial\Omega \cap \mathcal{N})} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{T}_r)} + \|D^2 V\|_{L^q(\mathcal{P}_{r,T})})
\end{aligned}$$

by (18) and $\partial\Omega_r^- \subset \mathcal{P}_{r,T}$. Further on, $\tilde{V} = V$ on $\mathcal{P}_{r,T+r}$, whence

$$\|D^2 V\|_{L^q(\mathcal{P}_{r,T+r})} \leq \|D^2 V\|_{L^q(\mathcal{P}_{r,T})} + \|D^2 \tilde{V}\|_{L^q(\Omega_r)}.$$

Therefore, setting $\zeta(T) := \|D^2 V\|_{L^q(\mathcal{P}_{r,T})}$ and $K := \|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{T}_r)}$, it follows from (24), (25) and Proposition 2 that

$$(26) \quad \zeta(T+r) \leq C(K + \zeta(T)) \quad \forall T \in (0, t^+ - t^-).$$

To get (23), we let m to be the least integer such that $T_{\max} \leq mr$ and iterate (26) in order to obtain

$$\begin{aligned}
\|D^2 V\|_{L^q(\mathcal{T}_r)} &= \|D^2 V\|_{L^q(\mathcal{P}_{r,T_{\max}})} = \zeta(T_{\max}) = \zeta(mr) = \zeta((m-1)r + r) \\
&\leq C(K + \zeta((m-1)r)) = C(K + \zeta((m-2)r + r)) \\
&\leq K(C + C^2) + C^2 \zeta((m-2)r) \\
&\vdots \\
&\leq K \sum_{j=1}^m C^j + C^m \zeta(0) = K \sum_{j=1}^m C^j
\end{aligned}$$

This proves (23). \square

Remark 6. It is important to note that the constant C in Proposition 5 depends on m through T_{\max} , and therefore on the point $x_0 \in \mathcal{E}$. Actually, that constant will have the very same value for each other point of \mathcal{E} lying on the same \mathbf{L} -trajectory as x_0 .

Moreover, if the *improving-of-integrability property* asserted in Propositions 4 and 5 holds on a set $S \subset \overline{\Omega}$ then it is guaranteed, on the base of (13), on any other set which can be reached from S along \mathbf{L} -trajectories.

To complete the proof of Lemma 3, we select a finite set $\{\mathcal{T}_r^j\}_{j=1}^N$ of neighbourhoods covering the compact \mathcal{E} , each of the type \mathcal{T}_r above with $r = r_0/2$, and such that $\mathcal{T} := \text{closure} \left(\bigcup_{j=1}^N \mathcal{T}_{r/2}^j \right) \subset \mathcal{N}''$ is a closed neighbourhood of \mathcal{E} in $\overline{\Omega}$. It is clear that Proposition 2 remains true with \mathcal{T} instead of \mathcal{N}' and then (7) rewrites into

$$(27) \quad \|u\|_{W^{2,q}(\Omega \setminus \mathcal{T})} \leq C (\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\varphi\|_{W^{1-1/q,q}(\partial\Omega)}).$$

The *improving-of-integrability* claimed in Lemma 3 then follows from (13), Proposition 5 and (27) (recall $U = u$ on $\mathcal{T}_{r/2}^j$). Similarly, (13), (27) and (23) yield

$$(28) \quad \begin{aligned} \|u\|_{W^{2,q}(\mathcal{N}'')} &\leq \|u\|_{W^{2,q}(\mathcal{T})} + \|u\|_{W^{2,q}(\mathcal{N}'' \setminus \mathcal{T})} \\ &\leq C(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})} \\ &\quad + \|u\|_{W^{1,q}(\mathcal{N})} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{N})}). \end{aligned}$$

Later on, $\mathcal{N} \setminus \mathcal{N}'' \subset \Omega \setminus \mathcal{N}'$ and

$$\begin{aligned} \|u\|_{W^{1,q}(\mathcal{N})} &\leq \|u\|_{W^{1,q}(\mathcal{N}'')} + \|u\|_{W^{1,q}(\mathcal{N} \setminus \mathcal{N}'')} \\ &\leq \varepsilon \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)(\|u\|_{L^q(\Omega)} + \|u\|_{W^{2,q}(\Omega \setminus \mathcal{N}')}) \end{aligned}$$

in view of the interpolation inequality for the $W^{2,q}(\mathcal{N}'')$ -norms with $\varepsilon > 0$ under control³. In the same manner,

$$\begin{aligned} \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{N})} &\leq \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{N}')} + \|\partial u / \partial \mathbf{L}\|_{W^{1,q}(\mathcal{N} \setminus \mathcal{N}')} \\ &\leq \varepsilon \|\partial u / \partial \mathbf{L}\|_{W^{2,q}(\mathcal{N}')} + C(\varepsilon)(\|\partial u / \partial \mathbf{L}\|_{L^q(\mathcal{N}')} + \|u\|_{W^{2,q}(\Omega \setminus \mathcal{N}')}), \end{aligned}$$

while

$$\|\partial u / \partial \mathbf{L}\|_{W^{2,q}(\mathcal{N}')} \leq C(\|u\|_{W^{2,q}(\mathcal{N}'')} + \|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})})$$

by means of the local *a priori* estimates ([3, Theorem 9.11]) for the problem (6).

A substitution of the above expressions into (28) and (7) give

$$\begin{aligned} \|u\|_{W^{2,q}(\mathcal{N}'')} &\leq C(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})} \\ &\quad + \varepsilon \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\|\partial u / \partial \mathbf{L}\|_{L^q(\mathcal{N}')}) \end{aligned}$$

whence, choosing $\varepsilon > 0$ small enough, we get

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N}')}).$$

Similarly, another application of the interpolation inequality yields

$$\|u\|_{W^{1,q}(\mathcal{N}')} \leq \|u\|_{W^{1,q}(\mathcal{N}'')} \leq \delta \|u\|_{W^{2,q}(\mathcal{N}'')} + C(\delta)\|u\|_{L^q(\mathcal{N}'')}$$

and thus

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega, \mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega, \mathcal{N})}).$$

for small $\delta > 0$. The proof of Lemma 3 is completed. \square

The statement of Theorem 1 follows from Proposition 2 and Lemma 3.

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³This requires some minimal smoothness of $\partial\mathcal{N}''$ and it is not restrictive to take it Lipschitz continuous at the very beginning.

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